Berry–Esseen bounds for periodic Langevin dynamics

How long until transport looks Gaussian?

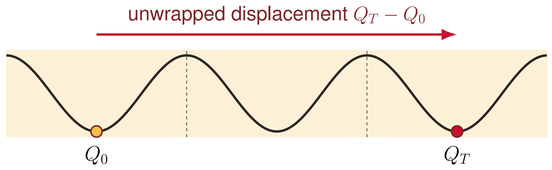
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The question

A particle crosses many cells of a periodic landscape. How accurately does its finite-time displacement follow a Gaussian law, and how does this depend on friction?



$V(q) = \cos(2\pi q)$. One cell repeats; the travelled distance accumulates.

We track the unwrapped position Q, then project onto a unit direction F and rescale:

$$Y_T^F = \frac{F^\top (Q_T - Q_0)}{\sqrt{T}}.$$

Two dynamics

OVERDAMPED · THEOREM 1

$$dq_t = -\nabla V(q_t)\,dt + \sqrt{2}\,dW_t$$

Smooth periodic potential, inverse temperature $\beta = 1$. The wrapped position starts in its invariant Gibbs law.

UNDERDAMPED · THEOREM 2

$$dq_t = p_t\,dt,$$
$$dp_t = -\nabla V(q_t)\,dt - \gamma p_t\,dt + \sqrt{2\gamma/\beta}\,dW_t$$

Position and momentum start from equilibrium; $\gamma > 0$ is the friction and $\beta > 0$ the inverse temperature.

What is measured

The Kolmogorov distance compares distribution functions,

$$d_K(X, Y) = \sup_{x \in \mathbb{R}} \big|\mathbb{P}(X \le x) - \mathbb{P}(Y \le x)\big|.$$

The Gaussian target uses the asymptotic displacement variance in direction F, $\sigma_F^2 = F^\top D F$.

Main results

THEOREM 1

Overdamped Gaussian approximation

$$d_K\big(\mathcal{L}(Y_T^F),\, \mathcal{N}(0, \sigma_F^2)\big) \le C_F\, T^{-1/2}$$

For every fixed unit direction F and all $T \ge 1$; C_F is independent of T.

THEOREM 2

Underdamped Gaussian approximation

$$d_K\big(\mathcal{L}(Y_T^{\gamma,F}),\, \mathcal{N}(0, \sigma_{\gamma,F}^2)\big) \le C_{\gamma,F}\, T^{-1/2}$$

For each fixed $\gamma > 0$ and unit F, for all $T \ge 1$. The asymptotic variance is positive; $C_{\gamma,F}$ may depend on friction and direction.

PROPOSITION 11.1

One-dimensional low-friction bound

$$d_K\big(\mathcal{L}(Y_T^\gamma),\, \mathcal{N}(0, \sigma_\gamma^2)\big) \le \frac{C}{\sqrt{\gamma T}}, \qquad \text{i.e.} \quad C_\gamma \le C\gamma^{-1/2}$$

For $0 < \gamma \le \gamma_0$ and $T \ge 1$, with equilibrium initial data. The same C works throughout this friction range.

The relevant observation scale is γT: time measured in friction relaxation units.

Scope. Fixed-time scalar projections. The statements are upper bounds; they do not give an exact error exponent or a quantitative theorem for the full path law.

Proof ideas

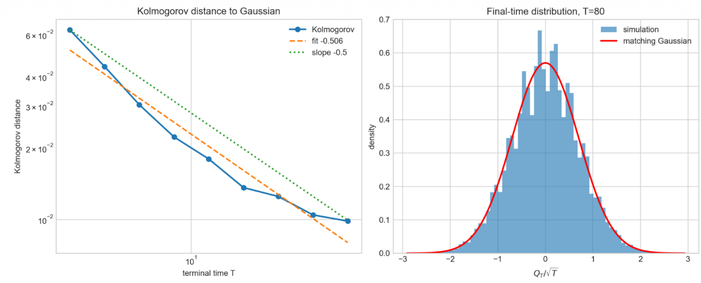
- **Theorem 1:** bounded corrector and a Markov-additive Berry–Esseen theorem.
- **Theorem 2:** a direct argument for the integrated velocity.
- **Proposition 11.1:** blocks on the friction relaxation scale, with uniform estimates,

$$h \asymp \gamma^{-1} \quad \Longrightarrow \quad n \asymp \gamma T.$$

Numerical illustrations

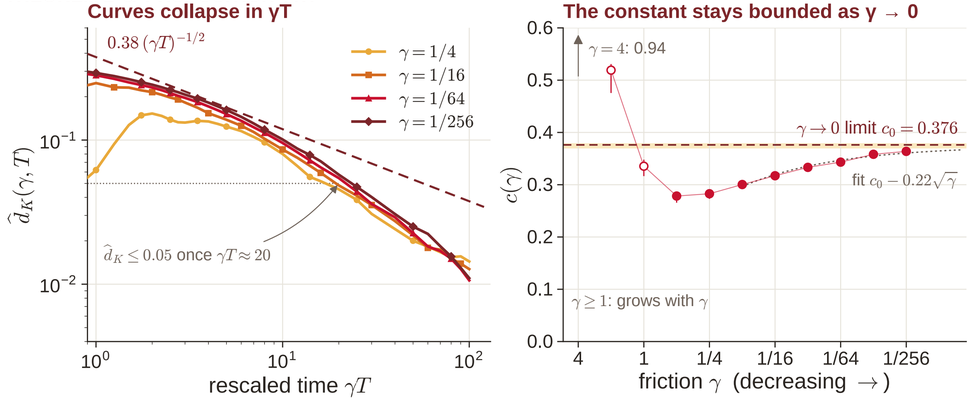
THEOREM 1

$V(q) = \cos(2\pi q)$, $\beta = 1$. Original Figure 1 from the manuscript.



Reported slope -0.506**.** At $T = 80$ the empirical Kolmogorov distance is 9.87×10^{-3}. Metropolized Euler–Maruyama, $\Delta t = 0.01$, 80,000 replicas, 1,000 burn-in steps; the plotted Gaussian uses the sample variance.

THEOREM 2 & PROPOSITION 11.1



In units of $1/\gamma$ the curves collapse. The smallest constant $c(\gamma) = \max_{T \ge 1} \sqrt{\gamma T}\, \widehat{d}_K$ stays below its $\gamma \to 0$ limit $c_0 \approx 0.38$, and grows for $\gamma \ge 1$, so the restriction $\gamma \le \gamma_0$ is needed.

$V(q) = \cos q$, $\beta = 1$; BAOAB, $\Delta t = 0.05$; 80,000 stationary trajectories per γ. c_0 from the averaged energy dynamics.

References

D. Ferré, L. Hervé, J. Ledoux (2012). Limit theorems for stationary Markov processes with L^2-spectral gap.

J. Roussel, G. Stoltz (2018). Spectral methods for Langevin dynamics and associated error estimates.

M. Hairer, G. A. Pavliotis (2008). From ballistic to diffusive behavior in periodic potentials.

Acknowledgements

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